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String Field Theory

LEONARDO RASTELLI

*Joseph Henry Laboratories,
Princeton University, Princeton, NJ 08544, USA*

ABSTRACT

This article is a concise review of covariant string field theory prepared for the Encyclopedia of Mathematical Physics, Elsevier (2006). Referencing follows the publisher's guidelines.

1. Introduction

String field theory (SFT) is the second-quantized approach to string theory. In the usual, first-quantized, formulation of string perturbation theory, one postulates a recipe for the string S-matrix in terms of a sum over two-dimensional worldsheets embedded in spacetime. Very schematically,

$$\langle\langle V_1(k_1) \dots V_n(k_n) \rangle\rangle = \sum_{topologies} g_s^{-\chi} \int [d\mu_\alpha] \langle V_1(k_1) \dots V_n(k_n) \rangle_{\{\mu_\alpha\}}. \quad (1.1)$$

Here the *lhs* stands for the S-matrix of the physical string states $\{V_a(k_a)\}$. The symbol $\langle \dots \rangle_{\{\mu_\alpha\}}$ denotes a correlation function on the 2d worldsheet, which is a punctured Riemann surface of Euler number χ and given moduli $\{\mu_\alpha\}$. In SFT, one aims to recover this standard prescription from the Feynman rules of a second-quantized *spacetime* action $S[\Phi]$. The string field Φ , the fundamental dynamical variable, can be thought of as an infinite dimensional array of spacetime fields $\{\phi^i(x^\mu)\}$, one field for each basis state in the Fock space of the first-quantized string.

The most straightforward way to construct $S[\Phi]$ uses the unitary light-cone gauge. Light-cone SFT is an almost immediate transcription of Mandelstam's light-cone diagrams in a second-quantized language. While often useful as a book-keeping device, light-cone SFT seems unlikely to represent a real improvement over the first-quantized approach. By contrast, from our experience in ordinary quantum field theory, we should expect Poincaré-covariant SFTs to give important insights into the issues of vacuum selection, background independence and the non-perturbative definition of string theory.

Covariant SFT actions are well-established for the open (Witten, 1986), closed (Zwiebach, 1993) and open/closed (Zwiebach, 1998) *bosonic* string. These theories are based on the BRST formalism, where the worldsheet variables include the *bc* ghosts introduced in gauge-fixing the worldsheet metric to the conformal gauge $g_{ab} \sim \delta_{ab}$. (An alternative approach (Hata et al.), based on covariantizing light-cone SFT, will not be described in this article.) Much less is presently known for the superstring: classical actions have been established for the NS sector of the open superstring (Berkovits, 2001) and for the heterotic string (Berkovits et al., 2004).

During the first period of intense activity in SFT (1985-1992), the covariant bosonic actions were constructed and shown to pass the basic test of reproducing the S-matrix (1.1) to each order in the perturbative expansion. The more

recent revival of the subject (since 1999) was triggered by the realization that SFT contains non-perturbative information as well: D-branes emerge as solitonic solutions of the classical equations of motion in open SFT. We can hope that the non-perturbative string dualities will also be understood in the framework of SFT, once covariant SFTs for the superstring are better developed.

In this article, we review the basic formalism of covariant SFT, using for illustration purposes the simplest model – cubic bosonic open SFT. We then briefly sketch the generalization to bosonic SFTs that include closed strings. Finally we turn to the subjects of classical solutions in open SFT and the physics of the open string tachyon.

2. Open Bosonic SFT

The standard formulation of string theory starts with the choice of an on-shell spacetime background where strings propagate. In the bosonic string, the closed string background is described by a conformal field theory of central charge 26 (the “matter” CFT). The total worldsheet CFT is the direct sum of this matter CFT and of the universal ghost CFT, of central charge -26 . To describe open strings, we must further specify boundary conditions for the string endpoints. The open string background is encoded in a Boundary CFT, a CFT defined in the upper half plane, with conformal boundary conditions on the real axis. (See the entry on BCFT in this Encyclopedia). In modern language, the choice of BCFT corresponds to specifying a D-brane state.

In classical open SFT, we fix the closed string background (the bulk CFT) and consider varying the D-brane configuration (the boundary conditions). To lowest order in g_s we can neglect the backreaction of the D-brane on the closed string fields, since this is a quantum effect from the open string viewpoint. Let us prepare the ground by recalling the standard σ -model philosophy. To describe off-shell open string configurations, we should allow for general (not necessarily conformal) boundary conditions. We can imagine to proceed as follows:

i) We choose an initial open string background, a reference BCFT that we shall call BCFT_0 . For example, a Dp brane in flat 26 dimensions (Neumann boundary conditions on $p + 1$ coordinates, Dirichlet on $25 - p$ coordinates).

ii) We then write a basis of boundary perturbations around *this* background. Taking for example BCFT_0 to be a D25 brane in flat space, the worldsheet action S_{WS} takes the schematic form

$$S_{WS} = \frac{1}{2\pi\alpha'} \int_{\mathbf{C}} \partial X_\mu \bar{\partial} X^\mu + \int_{\mathbf{R}} \tilde{T}(x^\mu) + \tilde{A}_\nu(x^\mu) \partial X^\nu + \tilde{B}_\nu(x^\mu) \partial^2 X^\nu + \dots \quad (2.1)$$

Here to the standard free bulk action (integrated over the whole complex plane \mathbf{C}) we have added a perturbation localized on the real axis \mathbf{R} . Notice that the basis of perturbations depends on the chosen BCFT_0 .

iii) We interpret the coefficients $\{\tilde{\phi}^i(x^\mu)\}$ of the perturbations as spacetime *fields*. (The tilde on $\tilde{\phi}^i(x)$ serves as a reminder that these fields are not quite the same as the fields $\phi^i(x)$ that will appear in the open SFT action). We are after a spacetime action $S[\{\tilde{\phi}^i\}]$ such that *solutions* of its classical equations of motion correspond to *conformal* boundary conditions:

$$\frac{\delta S}{\delta \phi^i} = 0 \text{ (spacetime)} \leftrightarrow \beta_i[\{\tilde{\phi}^j\}] = 0 \text{ (worldsheet)}. \quad (2.2)$$

We recognize in (2.1) the familiar open string tachyon $\tilde{T}(x)$ and gauge field $\tilde{A}_\mu(x)$, which are the lowest modes in an infinite tower of fields. Relevant perturbations on the worldsheet (with conformal dimension $h < 1$) correspond to tachyonic fields in spacetime ($m^2 < 0$), whereas marginal worldsheet perturbations ($h = 1$) give massless spacetime fields. To achieve a complete description, we must include all the higher massive open string modes as well, which correspond to non-renormalizable boundary perturbations ($h > 1$). In the traditional σ -model approach, this appears like a daunting task. The formalism of open SFT will automatically circumvent this difficulty.

2.1. The Open String Field

In covariant SFT the reparametrization ghosts play a crucial role. The ghost CFT consists of the Grassmann odd fields $b(z)$, $c(z)$, $\bar{b}(\bar{z})$, $\bar{c}(\bar{z})$, of dimensions $(2, 0)$, $(-1, 0)$, $(0, 2)$, $(0, -1)$. The boundary conditions on the real axis are $b = \bar{b}$, $c = \bar{c}$. The state space $\mathcal{H}_{\text{BCFT}_0}$ of the full matter + ghost Boundary CFT can be broken up into subspaces of definite ghost number,

$$\mathcal{H}_{\text{BCFT}_0} = \bigoplus_{G=-\infty}^{\infty} \mathcal{H}_{\text{BCFT}_0}^{(G)}. \quad (2.3)$$

We use conventions where the $\text{SL}(2, \mathbf{R})$ vacuum $|0\rangle$ carries zero ghost number, $G(|0\rangle) = 0$, while $G(c) = +1$ and $G(b) = -1$. As is familiar from the first-quantized treatment, physical open string states are identified with $G = +1$

cohomology classes of the BRST operator,

$$Q|V_{phys}\rangle = 0, \quad |V_{phys}\rangle \sim |V_{phys}\rangle + Q|\Lambda\rangle, \quad G(|V_{phys}\rangle) = +1, \quad (2.4)$$

where the nilpotent BRST operator Q has the standard expression

$$Q = \frac{1}{2\pi i} \oint (c T_{matter} + :bc\partial c:). \quad (2.5)$$

Though not *a priori* obvious, it turns out that the simplest form of the open SFT action is achieved by taking as the fundamental off-shell variable an *arbitrary* $G = +1$ element of the first-quantized Fock space,

$$|\Phi\rangle \in \mathcal{H}_{\text{BCFT}_0}^{(1)}. \quad (2.6)$$

By the usual state-operator correspondence of CFT, we can also represent $|\Phi\rangle$ as a local (boundary) vertex operator acting on the vacuum,

$$|\Phi\rangle = \Phi(0)|0\rangle. \quad (2.7)$$

The open string field $|\Phi\rangle$ is really an infinite-dimensional array of spacetime fields. We can make this transparent by expanding it as

$$|\Phi\rangle = \sum_i \int d^{p+1}k \, |\Phi_i(k)\rangle \phi^i(k^\mu), \quad (2.8)$$

where $\{|\Phi_i(k)\rangle\}$ is some convenient basis of $\mathcal{H}_{\text{BCFT}_0}^{(1)}$ that diagonalizes the momentum k_μ . The fields ϕ^i are a priori complex. This is remedied by imposing a suitable reality condition on the string field, which will be stated momentarily. Notice there are many more elements in $\{|\Phi_i(k)\rangle\}$ than in the physical subspace (the cohomology classes of Q). Some of the extra fields will turn out to be non-dynamical and could be integrated out, but at the price of making the OSFT action look much more complicated.

It is often useful to think of the string field in terms of its Schrödinger representation, that is, as a functional on the configuration space of open strings. Consider the unit half-disk in the upper-half plane, $D_H \equiv \{|z| \leq 1, \Im z \geq 0\}$, with the vertex operator $\Phi(0)$ inserted at the origin. Impose BCFT_0 open string boundary conditions for the fields $X(z, \bar{z})$ on the real axis (here $X(z, \bar{z})$ is a shorthand for all matter *and* ghost fields), and boundary conditions $X(\sigma) = X_b(\sigma)$

on the curved boundary of D_H , $z = \exp(i\sigma)$, $0 \leq \sigma \leq \pi$. The path-integral over $X(z, \bar{z})$ in the interior of the half-disk assigns a complex number to any given $X_b(\sigma)$, so we obtain a functional $\Phi[X_b(\sigma)]$. This is the Schrödinger wavefunction of the state $\Phi(0)|0\rangle$. Thus we can think of open string functionals $\Phi[X_b(\sigma)]$ as the fundamental variables of OSFT. This is as it should be: the first-quantized wavefunctions are promoted to dynamical fields in the second-quantized theory. Finally let us quote the reality condition for the string field, which takes a compact form in the Schrödinger representation:

$$\Phi[X^\mu(\sigma), b(\sigma), c(\sigma)] = \Phi^*[X^\mu(\pi - \sigma), b(\pi - \sigma), c(\pi - \sigma)], \quad (2.9)$$

where the superscript $*$ denotes complex conjugation.

2.2. The Classical Action

With all the ingredients in place, it is immediate to write the *quadratic* part of the OSFT action. The linearized equations of motion must reproduce the physical-state condition (2.4). This suggests

$$S \sim \langle \Phi | Q | \Phi \rangle. \quad (2.10)$$

Here $\langle | \rangle$ is the usual BPZ inner product of BCFT_0 , which is defined in terms of a two-point correlator on the disk, as we review below. The ghost anomaly implies that on disk we must have $G_{tot} = +3$, which happily is the case in (2.10). Moreover since the inner product is non-degenerate, variation of (2.10) gives

$$Q|\Phi\rangle = 0, \quad (2.11)$$

as desired. The equivalence relation $|V_{phys}\rangle \sim |V_{phys}\rangle + Q|\Lambda\rangle$ is interpreted in the second-quantized language as the *spacetime* gauge-invariance

$$\delta_\Lambda |\Phi\rangle = Q|\Lambda\rangle, \quad |\Lambda\rangle \in \mathcal{H}_{\text{BCFT}_0}^{(0)}, \quad (2.12)$$

valid for the general off-shell field. This equation is a very compact generalization of the linearized gauge-invariance for the massless gauge-field. Indeed, focusing on the level-zero components, $|\Phi\rangle \sim A_\mu(x) (c\partial X^\mu)(0)|0\rangle$ and $|\Lambda\rangle \sim \lambda(x)|0\rangle$, we find $\delta A_\mu(x) = \partial_\mu \lambda(x)$. It is then plausible to guess that the non-linear gauge-invariance should take the form

$$\delta_\Lambda |\Phi\rangle = Q|\Lambda\rangle + |\Phi\rangle * |\Lambda\rangle - |\Lambda\rangle * |\Phi\rangle, \quad (2.13)$$

where $*$ is some suitable product operation that conserves ghost number

$$* : \mathcal{H}_{\text{BCFT}_0}^{(n)} \otimes \mathcal{H}_{\text{BCFT}_0}^{(m)} \rightarrow \mathcal{H}_{\text{BCFT}_0}^{(n+m)}. \quad (2.14)$$

Based on a formal analogy with 3d non-abelian Chern-Simons theory, Witten proposed the cubic action

$$S = -\frac{1}{g_o^2} \left(\frac{1}{2} \langle \Phi | Q | \Phi \rangle + \frac{1}{3} \langle \Phi | \Phi * \Phi \rangle \right). \quad (2.15)$$

The string field $|\Phi\rangle$ is analogous to the Chern-Simons gauge-potential $A = A_i dx^i$, the $*$ product to the \wedge product of differential forms, Q to the exterior derivative d , and the ghost number G to the degree of the form. The analogy also suggests a number of algebraic identities:

$$\begin{aligned} Q^2 &= 0, \\ \langle Q A | B \rangle &= -(-1)^{G(A)} \langle A | Q B \rangle, \\ Q(A * B) &= (Q A) * B + (-1)^{G(A)} A * (Q B), \\ \langle A | B \rangle &= (-1)^{G(A)G(B)} \langle B | A \rangle, \\ \langle A | B * C \rangle &= \langle B | C * A \rangle, \\ A * (B * C) &= (A * B) * C. \end{aligned} \quad (2.16)$$

Note in particular the *associativity* of the $*$ -product. It is straightforward to check that this algebraic structure implies the gauge-invariance of the cubic action under (2.13). A $*$ -product satisfying all required formal properties can indeed be defined. The most intuitive presentation is in the functional language.

Given an open string curve $X(\sigma)$, $0 \leq \sigma \leq \pi$, we single out the string midpoint $\sigma = \pi/2$ and define the left and right “half-string” curves

$$\begin{aligned} X_L(\sigma) &\equiv X(\sigma) & \text{for } 0 \leq \sigma \leq \frac{\pi}{2}, \\ X_R(\sigma) &\equiv X(\pi - \sigma) & \text{for } \frac{\pi}{2} \leq \sigma \leq \pi. \end{aligned} \quad (2.17)$$

A functional $\Phi[X(\sigma)]$ can of course be regarded as a functional of the two half-strings, $\Phi[X] \rightarrow \Phi[X_L, X_R]$. We define

$$(\Phi_1 * \Phi_2)[X_L, X_R] \equiv \int [dY] \Phi_1[X_L, Y] \Phi_2[Y, X_R], \quad (2.18)$$

where $\int [dY]$ is meant as the functional integral over the space of half-strings $Y(\sigma)$, with $Y(\pi/2) = X_L(\pi/2) = X_R(\pi/2)$. The picture (Fig.1a) is that of two

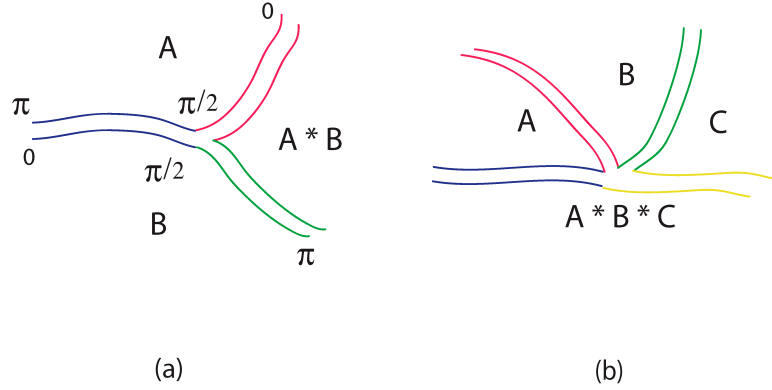


Figure 1: *Midpoint overlaps of open strings.*

open strings interacting (to form a single open string) if and only if the right half of the first string precisely overlaps with the left half of the second string. Associativity is transparent (Fig.1b).

We can now translate this formal construction in the precise CFT language. Very generally, an n -point vertex of open strings can be defined by specifying an n -punctured disk, that is, a disk with marked points on the boundary (punctures) and a choice of local coordinates around each puncture. Local coordinates are essential since we are dealing with off-shell open string states. The BPZ inner product (2pt vertex) is given by

$$\langle \Phi_1 | \Phi_2 \rangle \equiv \langle I \circ \Phi_1(0) \Phi_2(0) \rangle_{\text{UPH}}, \quad I(z) = -\frac{1}{z}. \quad (2.19)$$

The symbol $f \circ \Phi(0)$, where f is a complex map, means the conformal transform of $\Phi(0)$ by f . For example if Φ is a dimension d primary field, then $f \circ \Phi(0) = f'(0)^d \Phi(f(0))$. If Φ is non-primary the transformation rule will be more complicated and involve extra terms with higher derivatives of f . By performing the $\text{SL}(2, \mathbb{C})$ transformation

$$w = h(z) \equiv \frac{1 + iz}{1 - iz} \quad (2.20)$$

we can represent the 2pt vertex as a correlator on the unit disk $D = \{|w| \leq 1\}$,

$$\langle \Phi_1 | \Phi_2 \rangle = \langle f_1 \circ \Phi_1(0), f_2 \circ \Phi_2(0) \rangle_D, \quad f_1(z_1) = -h(z_1), \quad f_2(z_2) = h(z_2). \quad (2.21)$$

The vertex operators are inserted as $w = -1$ and $w = +1$ on D (see Fig.2a) and correspond to the two open strings at (Euclidean) worldsheet time $\tau = -\infty$ (we

take $z = \exp(i\sigma + \tau)$). The left half of D is the worldsheet of the first open string; the right half of D is the worldsheet of the second string. The two strings meet at $\tau = 0$ on the imaginary w axis. The 3pt Witten vertex is given by

$$\langle \Phi_1, \Phi_2, \Phi_3 \rangle \equiv \langle g_1 \circ \Phi_1(0) g_2 \circ \Phi_2(0) g_3 \circ \Phi_3(0) \rangle_D, \quad (2.22)$$

where

$$g_1(z_1) = e^{\frac{2\pi i}{3}} \left(\frac{1 + iz_1}{1 - iz_1} \right)^{\frac{2}{3}}, \quad g_2(z_2) = \left(\frac{1 + iz_2}{1 - iz_2} \right)^{\frac{2}{3}}, \quad g_3(z_3) = e^{-\frac{2\pi i}{3}} \left(\frac{1 + iz_3}{1 - iz_3} \right)^{\frac{2}{3}}. \quad (2.23)$$

The three punctured disk is depicted in Fig.2b, and describes the symmetric midpoint overlap of the three strings at $\tau = 0$. Finally the relation between the 3pt vertex and the $*$ -product is

$$\langle \Phi_1 | \Phi_2 * \Phi_3 \rangle \equiv \langle \Phi_1, \Phi_2, \Phi_3 \rangle. \quad (2.24)$$

Knowledge of the *rhs* in (2.24) for all Φ allows to reconstruct the $*$ -product. All formal properties (2.16) are easily shown to hold in the CFT language. This completes the definition of the OSFT action.

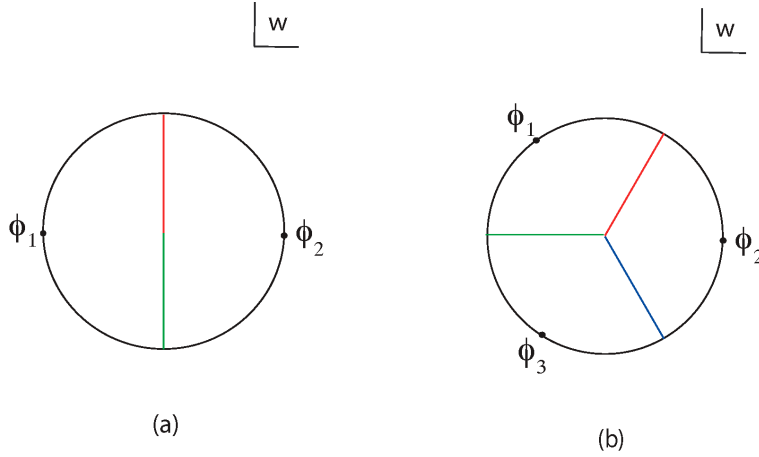


Figure 2: Representation of the quadratic and cubic vertices as 2- and 3-punctured unit disks.

Evaluation of the classical action is completely algorithmic and can be carried out for arbitrary massive states, with no fear of divergences, since in all required correlators the operators are inserted well apart from each other.

2.3. Quantization

Quantization is defined by the path-integral over the second-quantized string field. The first step is to deal with the gauge-invariance (2.13) of the classical action. The gauge symmetry is reducible: not all gauge-parameters $\Lambda^{(0)}$ (the superscript labels ghost number) lead to a gauge transformation. This is clear at the linearized level, indeed, if $\Lambda^{(0)} = Q\Lambda^{(-1)}$, then $\delta_{\Lambda^{(0)}}\Phi^{(1)} = Q^2\Lambda^{(0)} = 0$. Thus the set $\{\Lambda^{(0)}\}$ gives a redundant parametrization of the gauge group. Characterizing this redundancy is somewhat subtle, since fields of the form $\Lambda^{(-1)} = Q\Lambda^{(-2)}$ do not really lead to a redundancy in $\Lambda^{(0)}$, and so on, *ad infinitum*. It is clear that we need to introduce an infinite tower of (second-quantized) ghosts for ghosts.

The Batalin-Vilkovisky formalism is a powerful way to handle the problem. The basic object is the master action $S(\phi^s, \phi_s^*)$, which is a function of the “fields” ϕ^s and of the “antifields” ϕ_s^* . Each field is paired with a corresponding antifield of opposite Grassmanality. The master action must obey the boundary condition of reducing to the classical action when the antifields are set to zero. (Note that in general the set of fields ϕ^s will be larger than the set of fields ϕ^i that appear in the classical action). Independence of the S-matrix on the gauge-fixing procedure is equivalent to the BV *master equation*

$$\frac{1}{2}\{S, S\} = -\hbar \Delta S. \quad (2.25)$$

The antibracket $\{ , \}$ and the Δ operator are defined as

$$\{A, B\} \equiv \frac{\partial_r A}{\partial \phi^s} \frac{\partial_l B}{\partial \phi_s^*} - \frac{\partial_r A}{\partial \phi_s^*} \frac{\partial_l B}{\partial \phi^s}, \quad \Delta \equiv \frac{\partial_r}{\partial \phi^s} \frac{\partial_l}{\partial \phi_s^*}, \quad (2.26)$$

where ∂_l and ∂_r are derivatives from the left and from the right. It is often convenient to expand S in powers of \hbar , $S = S_0 + \hbar S_1 + \hbar^2 S_2 + \dots$, with

$$\{S_0, S_0\} = 0, \quad \{S_0, S_1\} + \{S_1, S_0\} = -2\hbar \Delta S_0, \quad \dots \quad (2.27)$$

With these definitions in place, we shall simply describe the answer, which is extremely elegant. In OSFT the full set of fields and antifields is packaged in a single string field $|\Phi\rangle$, of *unrestricted* ghost number. If we write

$$|\Phi\rangle = |\Phi_- \rangle + |\Phi_+ \rangle, \quad \text{with } G(\Phi_-) \leq 1 \text{ and } G(\Phi_+) \geq 2, \quad (2.28)$$

all the fields are contained in $|\Phi_-\rangle$ and all the antifields in $|\Phi_+\rangle$. To make the pairing explicit, we pick a basis $\{|\Phi_s\rangle\}$ of $\mathcal{H}_{\text{BCFT}_0}$, and define a conjugate basis $\{|\Phi_s^C\rangle\}$ by

$$\langle\Phi_r^C|\Phi_s\rangle = \delta_{rs} . \quad (2.29)$$

Clearly $G(\Phi_s^C) + G(\Phi_s) = 3$. Then

$$|\Phi_-\rangle = \sum_{G(\Phi_s)\leq 1} |\Phi_s\rangle \phi^s , \quad |\Phi_+\rangle = \sum_{G(\Phi_s)\leq 1} |\Phi_s^C\rangle \phi_s^* . \quad (2.30)$$

Basis states $|\Phi_s\rangle$ with even (odd) ghost number $G(\Phi_s)$ are defined to be Grassmann even (odd). The full string field $|\Phi\rangle$ is declared to be Grassmann odd. It follows that ϕ^s is Grassmann even (odd) for $G(\Phi_s)$ odd (even), and that the corresponding antifield ϕ_s^* has the opposite Grassmanality of ϕ^s , as it must be. With this understanding of $|\Phi\rangle$, the classical master action S_0 is identical in form to the Witten action (2.15)! The boundary condition is satisfied, indeed setting $|\Phi_+\rangle = 0$ the ghost number anomaly implies that only the terms with $G = +1$ survive. The equation $\{S_0, S_0\} = 0$ follows from straightforward manipulations using the algebraic identities (2.16). On the other hand, the issue of whether $\Delta S_0 = 0$, or whether instead quantum corrections are needed to satisfy full BV master equation, is more subtle and has never been fully resolved. The Δ operator receives singular contributions from the same region of moduli space responsible for the appearance of closed string poles, discussed below in section 2.5. (See (Thorn, 1989) for the classic statement of this issue). It seems possible to choose of basis in $\mathcal{H}_{\text{BCFT}_0}$ such that there are no quantum corrections to S_0 (Erler and Gross, 2004). In the following we shall derive the Feynman rules implied by S_0 alone.

2.4. SFT Diagrams and Minimal Area Metrics

Imposing the Siegel gauge condition $b_0\Phi = 0$, one finds the gauge-fixed action

$$S_{gf} = -\frac{1}{g_o^2} \left(\frac{1}{2} \langle\Phi| c_0 L_0 \Phi\rangle + \frac{1}{3} \langle\Phi|\Phi * \Phi\rangle + \langle\beta| b_0|\Phi\rangle \right) , \quad (2.31)$$

where β is a Lagrangian multiplier. The propagator reads

$$\frac{b_0}{L_0} = b_0 \int_0^\infty dT e^{-TL_0} . \quad (2.32)$$

Since L_0 is the first-quantized open string Hamiltonian, e^{-TL_0} is the operator that evolves the open string wave-functions $\Psi[X(\sigma)]$ by Euclidean worldsheet

time T . It can be visualized as a flat rectangular strip of “horizontal” width π and “vertical” height T . Each propagator comes with an antighost insertion

$$b_0 = \int_0^\pi b(\sigma) \quad (2.33)$$

integrated on a horizontal trajectory.

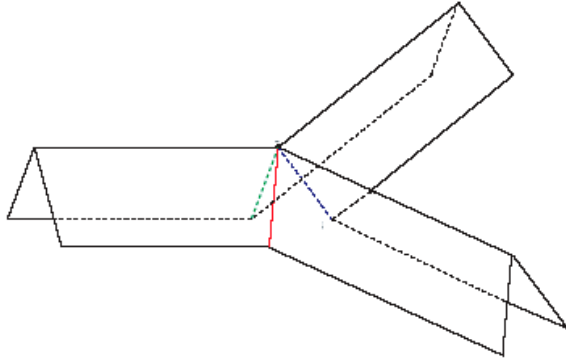


Figure 3: *The cubic vertex represented as the midpoint gluing of three strips.*

The only elementary interaction vertex is the midpoint three string overlap, visualized in Fig.3. We are instructed to draw all possible diagrams with given external legs (represented as semi-infinite strips), and to integrate over all Schwinger parameters $T_i \in [0, \infty)$ associated with the internal propagators. The claim is that this prescription reproduce precisely the first-quantized result (1.1). This follows if we can show that (i) the OSFT Feynman rules give a unique cover of the moduli space of open Riemann surfaces; (ii) the integration measure agrees with the measure $[d\mu_\alpha]$ in (1.1). This latter property holds because the antighost insertion (2.33) is precisely the one prescribed by the Polyakov formalism for integrating over the moduli T_i . To show point (i), we introduce the concept of minimal area metrics, which has proven very fruitful. (See Giddings et al. (1986), Giddings (1988) for the open SFT case and Zwiebach (1993, 1998) for the closed and open/closed SFTs). Quite generally, the Feynman rules of a SFT provide us with a cell decomposition of the appropriate moduli space of Riemann surfaces, a way to construct surfaces in terms of vertices and propagators. Given a Riemann surface (for fixed values of its complex moduli), the SFT must associate with it one and only one string diagram. The diagram has more structure than the Riemann surface: it defines a metric on it. In all known covariant SFTs, this is the metric of *minimal area* obeying suitable length conditions. Consider the following

Minimal Area problem for open SFT. Let \mathcal{R}_o be a Riemann surface with at least one boundary component and possibly punctures on the boundary. Find the (conformal) metric of minimal area on \mathcal{R}_o such that all non-trivial Jordan open curves have length greater or equal to π . (A curve is said to be non-trivial if it cannot be continuously shrunk to a point without crossing a puncture.)

An OSFT diagram (for fixed values of its T_i), defines a Riemann surface \mathcal{R}_o endowed with a metric solving this minimal area problem. This is the metric implicit in its picture: flat everywhere except at the conical singularities of defect angle $(n-2)\pi$ when n propagators meet symmetrically. (For $n=3$, these are the elementary cubic vertices; for $n>3$, they are effective vertices, obtained when propagators joining cubic vertices collapse to zero length.) It is not hard to see both that the length conditions are obeyed, and that the metric cannot be made smaller without violating a length condition. Conversely, any surface \mathcal{R}_o endowed with a minimal area metric, corresponds to an OSFT diagram. The idea is that the minimal area metric must have open geodesics (“horizontal trajectories”) of length π foliating the surface. The geodesics intersect on a set of measure zero – the “critical graph” where the propagators are glued. Bands of open geodesics of infinite height are the external legs of the diagram, while bands of finite height are the internal propagators.

The single cover of moduli space is then ensured by an existence and uniqueness theorem for metrics solving the minimal area problem for OSFT. These metrics are seen to arise from Jenkins-Strebel quadratic differentials. Existence shows that the Feynman rules of OSFT generate each Riemann surface \mathcal{R}_o at least once. Uniqueness shows there is no overcounting: since different diagrams correspond to different metrics (by inspection of their picture), no Riemann surface can be generated twice.

2.5. Closed Strings in OSFT

As is familiar, the open string S-matrix contains poles due to the exchange of on-shell open *and* closed strings. The closed string poles are present in non-planar loop amplitudes. We have seen that OSFT reproduces the standard S-matrix. Factorization over the open string poles is manifest, it corresponds to propagator lengths T_i going to infinity. Surprisingly, the closed string poles are also correctly reproduced, despite the fact that OSFT treats only the open strings as fundamental dynamical variables. In some sense, closed strings must be considered as derived objects in OSFT. Factorizing the amplitudes over the closed string poles one finds that on-shell closed string states can be represented, at least formally, as certain singular open string fields with $G = +2$, closely related to the (formal) identity string field. The picture is that of a folded open string, whose left and

right halves precisely overlap, with an extra closed string vertex operator inserted at the midpoint. The corresponding open/closed vertex is given by

$$\langle \Psi_{phys} | \Phi \rangle_{OC} \equiv \langle \Psi_{phys}(0) \mathcal{I} \circ \Phi(0) \rangle_D, \quad \mathcal{I} = \left(\frac{1+iz}{1-iz} \right)^2, \quad (2.34)$$

and describes the coupling to the open string field of a non-dynamical, *on-shell* closed string $|\Psi_{phys}\rangle$. It is possible to add this open/closed vertex to the OSFT action. Remarkably, the resulting Feynman rules give a single cover of the moduli space of Riemann surfaces with at least one boundary, with open *and* closed punctures. This is shown using the same minimal area problem as above, but now allowing for surfaces with closed punctures as well.

We should finally mention that the structure of OSFT emerges frequently in topological string theory, in contexts where open/closed duality plays a central role. Two examples are the interpretation of Chern-Simons theory as the OSFT for the A-model on the conifold, and the interpretation of the Kontsevich matrix integral for topological gravity as the OSFT on FZZT branes in $(2,1)$ minimal string theory.

3. Closed Bosonic SFT

The generalization to covariant closed SFT is non-trivial, essentially because the requisite closed string decomposition of moduli space is much more complicated.

The free theory parallels the open case, with a minor complication in the treatment of the CFT zero modes. The closed string field is taken to live in a *subspace* of the matter + ghost state space, $|\Psi\rangle \in \tilde{\mathcal{H}}_{\text{CFT}_0}$, where the tilde means that we impose the subsidiary conditions

$$b_0^- |\Psi\rangle = L_0^- |\Psi\rangle = 0, \quad b_0^- \equiv b_0 - \bar{b}_0, L_0^- \equiv L_0 - \bar{L}_0. \quad (3.1)$$

In the classical theory the string field carries ghost number $G = +2$, since it is the off-shell extension of the familiar closed string physical states, and the quadratic action reads

$$S \sim \langle \Psi, Q_c \Psi \rangle. \quad (3.2)$$

Here Q_c is the usual closed BRST operator. The inner product \langle, \rangle is defined in terms of the BPZ inner product, with an extra insertion of $c_0^- \equiv c_0 - \bar{c}_0$,

$$\langle A, B \rangle \equiv \langle A | c_0^- | B \rangle. \quad (3.3)$$

In (3.2) $G_{top} = +6$, as it should be. Without the extra ghost insertion and the subsidiary conditions (3.1) it would not be possible to write a quadratic action. The linearized equations of motion and gauge invariance,

$$Q_c|\Psi\rangle = 0, \quad |\Psi\rangle \sim |\Psi\rangle + Q_c|\Lambda\rangle, \quad |\Lambda\rangle \in \tilde{\mathcal{H}}_{\text{CFT}_0}^{(1)}, \quad (3.4)$$

give the expected cohomological problem. The fact that the cohomology is computed in the *semi-relative* complex, $b_0^-|\Psi\rangle = b_0^-|\Lambda\rangle = 0$, well-known from the operator formalism of the first-quantized theory, is recovered naturally in the second-quantized treatment.

The interacting action is constructed iteratively, by demanding that the resulting Feynman rules give a (unique) cover of moduli space. This requires the introduction of infinitely many elementary string vertices $\mathcal{V}_{g,n}$, where n is the number of closed string punctures and g the genus. This decomposition of moduli space is more intricate than the decomposition that arises in OSFT, but is in fact analogous to it, when characterized in terms of the following

Minimal Area problem for closed SFT. Let \mathcal{R}_c be a closed Riemann surface, possibly with punctures. Find the (conformal) metric of minimal area on \mathcal{R} such that all non-trivial Jordan closed curves have length greater or equal to 2π .

The minimal area metric induces a foliation of \mathcal{R}_c by closed geodesics of length 2π . In the classical theory ($g = 0$), the minimal area metrics arise from Jenkins-Strebel quadratic differentials (as in the open case), and geodesics intersect on a measure zero set. For $g > 0$ however there can be foliation bands of geodesics that cross. By staring at the foliation we can break up the surface into vertices and propagators. In correspondence with each puncture there is a band of infinite height, a flat semi-infinite cylinder of circumference 2π , which we identify as an external leg of the diagram. We mark a closed geodesic on each semi-infinite cylinder, at a distance π from its boundary. Bands of finite height (internal bands not associated to punctures) correspond to propagators if their height is greater than 2π , otherwise they are considered part of an elementary vertex. Along any internal cylinder of height greater than 2π , we mark two closed geodesics, at a distance π from the boundary of the cylinder. If we now cut open all the marked curves, the surface decomposes into a number of semi-infinite cylinders (external legs), finite cylinders (internal propagators) and surfaces with boundaries (elementary interactions). Each elementary interaction of genus g and with n boundaries is an element of $\mathcal{V}_{g,n}$. A crucial point of this construction is that we took care of leaving a “stub” of length π attached to each boundary. Stubs ensure that sewing of surfaces preserves the length condition on the metric (no closed curve shorter than 2π).

These geometric data can be translated into an iterative algebraic construction of the full quantum action $S[\Psi]$. The $\mathcal{V}_{g,n}$ satisfy geometric recursion relations whose algebraic counterpart is the quantum BV master equation for $S[\Psi]$. Remarkably, the singularities of the Δ operator encountered in OSFT are absent here, precisely because of the presence of the stubs. We refer to (Zwiebach, 1993) for a complete discussion of closed SFT.

3.1. Open/closed SFT

There is also a covariant SFT that includes both open and closed strings as fundamental variables. The Feynman rules arise from the natural

Minimal Area problem for open/closed SFT. Let \mathcal{R}_{oc} be a Riemann surface, with or without boundaries, possibly with open and closed punctures. Find the (conformal) metric of minimal area on \mathcal{R}_{oc} such that all non-trivial Jordan open curves have length greater or equal to $l_o = \pi$, and all non-trivial Jordan closed curves have length greater or equal to $l_c = 2\pi$.

The surface \mathcal{R}_{oc} is decomposed in terms of elementary vertices $\mathcal{V}_{b,m}^{g,n}$ (of genus g , b boundary components, n closed string punctures and m open string punctures) joined by open and closed propagators. Degenerations of the surface correspond always to propagators becoming of infinite length – factorization is manifest both in the open and in the closed channel.

The SFT described in section 2.5 (Witten OSFT augmented with the single open/closed vertex (2.34)) corresponds to taking $l_o = \pi$ and $l_c = 0$. Varying $l_c \in [0, 2\pi]$, we find a whole family of interpolating SFTs. This construction clarifies the special status of the Witten theory: moduli space is covered by a single cubic open overlap vertex, with no need to introduce dynamical closed strings, but at the price of a somewhat singular formulation.

4. Classical Solutions in Open SFT

In the present formulation of SFT, a background (a classical solution of string theory) must be chosen from the outset. The very definition of the string field requires to specify a (B)CFT₀. Intuitively, the string field lives in the “tangent” to theory space at a specific point – where “theory space” is some notion of a “space of 2d (boundary) quantum field theories”, not necessarily conformal. In the early 1990’s independence on the choice of background was demonstrated for *infinitesimal* deformations: the SFT actions written using neighboring (B)CFTs are indeed related by a field redefinition. In recent years it has become apparent that at least the *open* string field reaches out to open string backgrounds a *finite*

distance away – possibly covering the whole of theory space. (Classical solutions of closed SFT are beginning to be investigated at the time of this writing (2005)).

The OSFT action written using BCFT_0 data is just the full worldvolume action of the D-brane with BCFT_0 boundary conditions. Which classical solutions should we expect in this OSFT? In the bosonic string, Dp branes carry no conserved charge and are unstable. This instability is reflected in the presence of a mode with $m^2 = -1/\alpha'$, the open string tachyon $T(x^\mu)$, $\mu = 0, \dots, p$. From this physical picture, Sen argued that:

(i) The tachyon potential, obtained by eliminating the higher modes of the string field by their equations of motion, must admit a local minimum corresponding to the vacuum with no D-brane at all (henceforth, the *tachyon vacuum*, $T(x^\mu) = T_0$).

(ii) The value of the potential at T_0 (measured with respect to the BCFT_0 point $T = 0$) must be exactly equal to minus the tension of the brane with BCFT_0 boundary conditions.

(iii) There must be no perturbative open string excitations around the tachyon vacuum.

(iv) There must be space-dependent “lump” solutions corresponding to lower-dimensional branes. For example a lump localized along one worldvolume direction, say x^1 , such that $T(x^1) \rightarrow T_0$ as $x^1 \rightarrow \pm\infty$, is identified with a $D(p-1)$ brane.

Sen’s conjectures have all been verified in OSFT. (See (Sen, 2004) and (Taylor and Zwiebach, 2003) for reviews). The deceptively simple-looking equations of motion (in Siegel gauge)

$$L_0|\Phi\rangle + b_0(|\Phi\rangle * |\Phi\rangle) = 0, \quad (4.1)$$

are really an infinite system of coupled equations, and no analytic solutions are known. Turning on a vev for the tachyon drives into condensation an infinite tower of modes. Fortunately the approximation technique of “level truncation” is surprisingly effective. The string field is restricted to modes with an L_0 eigenvalue smaller than a prescribed maximal level L . For any finite L , the truncated OSFT contains a finite number of fields and numerical computations are possible. Numerical results for various classical solutions converge quite rapidly as the level L is increased.

The most important solution is the string field $|\mathcal{T}\rangle$ that corresponds to the tachyon vacuum. A remarkable feature of $|\mathcal{T}\rangle$ is *universality*: it can be written as a linear combination of modes obtained by acting on the tachyon $c_1|0\rangle$ with

ghost oscillators and matter Virasoro operators,

$$|\mathcal{T}\rangle = T_0 c_1 |0\rangle + u L_{-2}^m c_1 |0\rangle + v c_{-1} |0\rangle + \dots$$

This implies that the properties of $|\mathcal{T}\rangle$ are independent of any detail of BCFT_0 , since all computations involving $|\mathcal{T}\rangle$ can be reduced to purely combinatorial manipulations involving the ghosts and the Virasoro algebra. The numerical results strongly confirm Sen’s conjectures, and indicate that the tachyon vacuum is located at a non-singular point in configuration space. Numerical solutions describing lower-dimensional branes and exactly marginal deformations are also available. For example, the full family of solutions interpolating between a D1 and a D0 brane at the self-dual radius has been found. There is increasing evidence that the open string field provides a faithful map of the open string landscape.

4.1. Vacuum SFT: D-branes as Projectors

In the absence of a closed-form expression for $|\mathcal{T}\rangle$, we are led to guesswork. When expanded around $|\mathcal{T}\rangle$, the OSFT is still cubic, only with a different kinetic term \mathcal{Q} ,

$$S = -\kappa_0 \left[\frac{1}{2} \langle \Phi | \mathcal{Q} | \Phi \rangle + \frac{1}{3} \langle \Phi | \Phi * \Phi \rangle \right]. \quad (4.2)$$

The operator \mathcal{Q} must obey all the formal properties (2.16), must be universal (constructed from ghosts and matter Virasoro operators), and must have trivial cohomology at $G = +1$. Another constraint comes from requiring that (4.2) admits classical solutions in Siegel gauge. The choice

$$\mathcal{Q} = \frac{1}{2i} (c(i) - \bar{c}(i)) = c_0 - (c_2 + c_{-2}) + (c_4 + c_{-4}) - \dots, \quad (4.3)$$

satisfies all these requirements. The conjecture (Rastelli, Sen and Zwiebach, 2001) is that by a field redefinition, the kinetic term around the tachyon vacuum can be cast into this form. This “purely ghost” \mathcal{Q} is somewhat singular (it acts at the delicate string midpoint), and presumably should be regarded as the leading term of a more complicated operator that includes matter pieces as well. The normalization constant κ_0 is formally infinite. Nevertheless, a regulator (for example level truncation) can be introduced, and physical observables are finite and independent of the regulator. The Vacuum SFT (4.2, 4.3) appears to capture the correct physics, at least at the classical level. Taking a matter/ghost factorized ansatz

$$|\Phi_g\rangle \otimes |\Phi_m\rangle, \quad (4.4)$$

and assuming that the ghost part is universal for all D-branes solutions, the equations of motion reduce to following equations for the matter part,

$$|\Phi_m\rangle * |\Phi_m\rangle = |\Phi_m\rangle. \quad (4.5)$$

A solution $|\Phi_m\rangle$ can be regarded as a *projector* acting in “half-string space”. Recall that the $*$ -product looks formally like a matrix multiplication (2.18): the matrices are the string fields, whose “indices” run over the half-string curves. These projector equations have been exactly solved by many different techniques (see (Rastelli, 2004) for a review). In particular there is a general BCFT construction that shows that one can obtain solutions corresponding to *any* D-brane configuration, including *multiple* branes – the rank of the projector is the number of branes. A rank one projector corresponds to an open string functional which is left/right *split*, $\Phi[X(\sigma)] = F_L(X_L)F_R(X_R)$. There is also clear analogy between these solutions and the soliton solutions of non-commutative field theory. The analogy can be made sharper using a formalism that re-writes the open string $*$ -product as the tensor product of infinitely many Moyal products. (See (Bars, 2002) and references therein).

It is unclear whether or not multiple brane solutions (should) exist in the original OSFT – they are yet to be found in level truncation. Understanding this and other issues, like the precise role of closed strings in the quantum theory (section 2.5) seems to require a precise characterization of the allowed space of open string functionals. In principle, the path-integral over such functionals would define the theory at the full non-perturbative level. This remains a challenge for the future.

Note added in proof:

Very recently, M. Schnabl, building on previous work on star algebra projectors and related surface states (Rastelli (2004) and references therein) was able to find the exact solution for the universal tachyon condensate in OSFT. This breakthrough is likely to lead to rapid new developments in SFT.

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